An application of dilation theory to hyponormal operators

By BÉLA SZ.-NAGY in Szeged and CIPRIAN FOIAŞ in Bucharest

Dedicated to our friend Professor K. Tandori on his 50th anniversary

Among the classes of operators extending the class of *normal* operators one of the most interesting is the class of *hyponormal* operators, i.e. operators H such that

$$H^*H \geqq HH^* \tag{1}$$

(see [1], § 160). The aim of this Note is to exhibit a connection between these two classes, by applying the theory of isometric dilations of contractions (see [2], Chapter II). We shall prove namely the following

Theorem. *For every hyponormal operator H on a Hilbert space $\mathfrak{H}$ there exist a normal operator N and a unitary operator U on some Hilbert space $\mathfrak{G}$, and a contraction X of $\mathfrak{H}$ into $\mathfrak{G}$, such that*

(a) $H=X^*NX$,

(b) $\|N\|=\|H\|$,

(c) $NU=UN=N^*$,

(d) $\|X^*Ug\|\leqq\|X^*g\|$ *for all* $g\in\mathfrak{G}$,

(e) *the manifolds* $\mathfrak{L}_n=U^nX\mathfrak{H}$ $(n=0,1,\ldots)$ *form a non-decreasing sequence and span* $\mathfrak{G}$,

(f) *for any complex scalars* α, β,

$$\sigma(\alpha N+\beta N^*)\subset\sigma_l(\alpha H+\beta H^*) \quad (\sigma_l\text{: “left spectrum”}).$$

Remark. This theorem has a rather trivial converse. Indeed, suppose H is any operator on $\mathfrak{H}$ which derives from operators N, U on a space $\mathfrak{G}$ and from a contraction $X\colon \mathfrak{H}\to\mathfrak{G}$ in such a way that conditions (a), (c) and (d) hold. Then H is hyponormal. Indeed, we have for $h\in\mathfrak{H}$

$$\|H^*h\| \stackrel{a}{=} \|X^*N^*Xh\| \stackrel{c}{=} \|X^*UNXh\| \stackrel{d}{\leqq} \|X^*NXh\| \stackrel{a}{=} \|Hh\|,$$

and the inequality $\|H^*h\|\leqq\|Hh\|$ for all $h\in\mathfrak{H}$ is equivalent to (1).

1. From the assumption that H is a hyponormal operator on $\mathfrak{H}$ it immediately follows that there exists a contraction T on $\mathfrak{H}$ such that

$$H^* = TH; \tag{2}$$

T is uniquely determined by H if we require that it be 0 on the orthogonal complement of the range of H.

From (2) we deduce $H=(TH)^*=H^*T^*=THT^*$, and hence,

$$H = T^n H T^{*n} \quad (n = 0, 1, \dots). \tag{3}$$

Let V be the minimal isometric dilation of the contraction T, acting on a Hilbert space $\mathfrak{K}(\supset\mathfrak{H})$. Then we have

$$T^n P_{\mathfrak{H}} = P_{\mathfrak{H}} V^n \quad (n = 0, 1, \dots), \tag{4}$$

where $P_{\mathfrak{H}}$ denotes orthogonal projection from $\mathfrak{K}$ onto $\mathfrak{H}$. The minimality property means that

$$\mathfrak{K} = \bigvee_{n\geqq 0} V^n \mathfrak{H}. \tag{5}$$

It is well-known that the subspace

$$\mathfrak{G} = \bigcap_{n\geqq 0} V^n \mathfrak{K} \tag{6}$$

reduces V to its *unitary* part

$$U = V|\mathfrak{G}. \tag{7}$$

Denote by $P_{\mathfrak{G}}$ the orthogonal projection $\mathfrak{K}\to\mathfrak{G}$. Since $P_{\mathfrak{G}}$ commutes with V we infer from (5) and (7) that

$$\mathfrak{G} = P_{\mathfrak{G}}\mathfrak{K} = \bigvee_{n\geqq 0} P_{\mathfrak{G}} V^n \mathfrak{H} = \bigvee_{n\geqq 0} U^n X \mathfrak{H}, \tag{8}$$

where we set

$$X = P_{\mathfrak{G}}|\mathfrak{H}. \tag{9}$$

Now the following representation is valid for X (see [2] Chapter II):

$$X = \lim_{n\to\infty} V^n T^{*n}, \tag{10}$$

whence

$$V^m X T^{*m} = X \quad (m = 0, 1, \dots). \tag{11}$$

This implies that the linear manifolds

$$\mathfrak{L}_n = V^n X \mathfrak{H} = U^n X \mathfrak{H} \quad (n = 0, 1, \dots) \tag{12}$$

form a non-decreasing sequence; their union

$$\mathfrak{L} = \bigcup_{n\geqq 0} \mathfrak{L}_n$$

is, on account of (8), dense in $\mathfrak{G}$.

2. Next we prove that the (strong) limit

$$X_H = \lim_{n\to\infty} V^n H T^{*n} \tag{13}$$

also exists. Indeed, from (3) and (4) we deduce, for $n \geqq m \geqq 0$ and $h \in \mathfrak{H}$:

$$(V^n HT^{*n}h, V^m HT^{*m}h) = (V^{n-m}HT^{*n}h, HT^{*m}h) = (T^{n-m}HT^{*n}h, HT^{*m}h) =$$
$$= (T^{n-m}HT^{*n-m}T^{*m}h, HT^{*m}h) = (HT^{*m}h, HT^{*m}h),$$

and hence,

$$\|V^n HT^{*n}h - V^m HT^{*m}h\|^2 = \|HT^{*n}h\|^2 - \|HT^{*m}h\|^2;$$

the assertion follows if we observe that the sequence $\|HT^{*n}h\|$ $(n=0, 1, \ldots)$ is convergent. This sequence is namely bounded by $\|H\| \cdot \|h\|$, and non-decreasing by the above relation.

In analogy to (11) we obtain from (13)

$$V^m X_H T^{*m} = X_H \quad (m = 0, 1, \ldots). \tag{14}$$

Note that (14) implies, in particular,

$$X_H \mathfrak{H} \subset V^m X_H \mathfrak{H} \subset V^m \mathfrak{K} \quad (m = 0, 1, \ldots)$$

and therefore,

$$X_H \mathfrak{H} \subset \mathfrak{G}. \tag{15}$$

3. Now we are ready to define the operator N. Let $U^{n_i} X h_i$ $(i=1, \ldots, s)$ be elements of the linear manifold $\mathfrak{L}$ and form the sums

$$\varphi = \sum_i U^{n_i} X h_i \quad \text{and} \quad \varphi_H = \sum_i U^{n_i} X_H h_i \quad (\in \mathfrak{G}, \text{ by } (15)).$$

Use (10) and (14) to obtain

$$\varphi = \lim_{r\to\infty} \sum_i V^{n_i} V^{r-n_i} T^{*r-n_i} h_i = \lim_{r\to\infty} V^r \sum_i T^{*r-n_i} h_i$$

and

$$\varphi_H = \lim_{r\to\infty} \sum_i V^{n_i} V^{r-n_i} H T^{*r-n_i} h_i = \lim_{r\to\infty} V^r H \sum_i T^{*r-n_i} h_i.$$

Hence we infer

$$\|\varphi_H\| \leqq \|H\| \cdot \|\varphi\|.$$

Because $\mathfrak{L}$ is dense in $\mathfrak{G}$ we conclude that there is an operator N on $\mathfrak{G}$, uniquely determined by the conditions

$$N\colon\ U^n X h \to U^n X_H h \quad (h \in \mathfrak{H};\ n = 0, 1, \ldots), \tag{16}$$

and we have

$$\|N\| \leqq \|H\|. \tag{17}$$

As $NU \cdot U^n X h = NU^{n+1} X h = U^{n+1} X_H h = U \cdot NU^n X h$, we have

$$NU = UN. \tag{18}$$

Moreover, using (3) we obtain for $h, h' \in \mathfrak{H}$; $s=0, 1, \dots$, and $n=0, 1, \dots$:

$$(Hh, T^{*s}h') = (T^n HT^{*n}h, T^{*s}h') = (HT^{*n}h, T^{*s+n}h') =$$
$$= (V^s V^n HT^{*n}h, V^{s+n}T^{*s+n}h');$$

letting $n \to \infty$ this implies

$$(T^s Hh, h') = (U^s X_H h, Xh') \quad (h, h' \in \mathfrak{H};\ s = 0, 1, \dots). \tag{19}$$

Consider any two elements of $\mathfrak{L}$; we may assume that both are in the same manifold $\mathfrak{L}_n$, thus can be written as

$$g = U^n Xh, \quad g' = U^n Xh'.$$

Using (2), (16) and (19) (for $s=1$, $s=0$) we get

$$(NUg, g') = (NU^{n+1}Xh, U^n Xh') = (U^{n+1}X_H h, U^n Xh') = (UX_H h, Xh') = (THh, h') =$$
$$= (H^*h, h') = (h, Hh') = (Xh, X_H h') = (U^n Xh, U^n X_H h') = (g, Ng').$$

This proves that

$$N^* = NU. \tag{20}$$

Relations (18) and (20) imply in particular that N is *normal.*

Again from (16), (19) ($s=0$) we deduce

$$(Hh, h') = (X_H h, Xh') = (NXh, Xh') = (X^* NXh, h');$$

whence,

$$H = X^* NX. \tag{21}$$

From the definition (9) of X we readily obtain that

$$X^* = P_{\mathfrak{H}} | \mathfrak{G}. \tag{22}$$

As both X and X^* are *contractions*, a comparison of (17) with (21) yields:

$$\|N\| = \|H\|. \tag{23}$$

Next observe that (4) implies $\|P_{\mathfrak{H}} Vg\| = \|TP_{\mathfrak{H}} g\| \leqq \|P_{\mathfrak{H}} g\|$ for all $g \in \mathfrak{K}$, and in particular for $g \in \mathfrak{G}$, so we have by (22)

$$\|X^* Ug\| \leqq \|X^* g\| \quad \text{for} \quad g \in \mathfrak{G}. \tag{24}$$

4. On account of the definitions (16) and (13) we have

$$NXh = \lim_{n\to\infty} V^n HT^{*n}h \quad (h \in \mathfrak{H}).$$

Hence, using (18), (20), and (2), we deduce

$$N^* Xh = UNXh = \lim_{n\to\infty} V^{n+1}HT^{*n}h = \lim_{n\to\infty} V^{n+1}(H^*T^*)T^{*n}h =$$
$$= \lim_{n\to\infty} V^{n+1}H^* T^{*n+1}h.$$

If we also recall (10) we conclude that for any complex scalars α, β, λ:

(25) $$(\alpha N+\beta N^*-\lambda I)Xh = \lim_{n\to\infty} V^n(\alpha H+\beta H^*-\lambda I)T^{*n}h.$$

Hence, setting $N_{\alpha\beta}=\alpha N+\beta N^*$ and $H_{\alpha\beta}=\alpha H+\beta H^*$, we have

(26) $$\|(N_{\alpha\beta}-\lambda I)Xh\| = \lim_{n\to\infty} \|(H_{\alpha\beta}-\lambda I)T^{*n}h\|.$$

Suppose λ is a point of "regular type" for $H_{\alpha\beta}$, i.e. that the inequality

(27) $$\|(H_{\alpha\beta}-\lambda I)h\| \geqq \varepsilon\|h\|$$

holds for some $\varepsilon>0$ and for all $h\in\mathfrak{H}$. Then by (26) and (10) we have

$$\|(N_{\alpha\beta}-\lambda I)Xh\| \geqq \varepsilon\cdot\lim_{n\to\infty}\|T^{*n}h\| = \varepsilon\cdot\|Xh\|,$$

and as U commutes with N and N^* we have

$$\|(N_{\alpha\beta}-\lambda I)U^nXh\| \geqq \varepsilon\|U^nXh\| \quad (h\in\mathfrak{H};\ n=0,1,\dots)$$

as well. As the manifolds $\mathfrak{L}_n$ are nested and span $\mathfrak{G}$ we conclude that

(28) $$\|(N_{\alpha\beta}-\lambda I)g\| \geqq \varepsilon\|g\| \quad \text{for all} \quad g\in\mathfrak{G}.$$

This implies that λ is in the resolvent set of the (normal) operator $N_{\alpha\beta}$.

Passing to the complement of the set of the above points λ we conclude that

$$\sigma(N_{\alpha\beta})\subset\sigma_l(H_{\alpha\beta}).$$

The proof of the Theorem is complete.

References

[1] P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand (Princeton—Toronto—London, 1967).

[2] B. Sz.-Nagy—C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, Akadémiai Kiadó — North Holland (Budapest—Amsterdam, 1970).

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